APROXIMAÇÃO POLINOMIAL

$f(x) = 2x^2 - 1$

$f(2) = 2 \cdot 2^2 - 1 = 7.$

$f(x) = e^x \qquad f(2),\ g(2)$

$g(x) = \operatorname{sen} x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$\therefore\ f(x+h) \approx f(x) + h f'(x)$$

$x = 0$

$$e^h \approx 1 + h$$

$$\operatorname{sen} h \approx 0 + h$$

$$\frac{E(h)}{h} = \left[\frac{f(x+h) - [f(x) + h f'(x)]}{h}\right]$$

$$\frac{E(h)}{h} = \frac{f(x+h) - f(x)}{h} - f'(x)$$

$$\boxed{\lim_{h \to 0} \frac{E(h)}{h} = 0}$$

a fixo

$$f(x) = \underbrace{f(a) + (x-a) \cdot f'(a)}_{p(x)} + E(x)$$

$$p(x) = f(a) + (x-a) f'(a)$$

grau 1 $\qquad p'(x) = f'(a)$

$$f(x) = p(x) + E(x)$$

$$\begin{cases} p(a) = f(a) \\ p'(a) = f'(a) \end{cases}$$

$$f(x) = \underbrace{p(x)}_{\text{grau } n} + E(h)$$

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$$

$$\left\{\begin{array}{l} p(a) = f(a) \\ p'(a) = f'(a) \\ p''(a) = f''(a) \\ \vdots \\ p^{(n)}(a) = f^{(n)}(a) \end{array}\right\} \Rightarrow$$

$$p(a) = f(a)$$

$$\| \quad c_0 + \cancel{c_1(a-a) + \dots + c_n(a-a)^n}$$

$$\therefore \boxed{c_0 = f(a)}$$

$$p'(x) = c_1 + 2c_2(x-a) + \dots + nc_n(x-a)^{n-1}$$

$$p'(a) = c_1 + \cancel{2c_2(a-a) + \dots + nc_n(a-a)^{n-1}}$$

$$f'(a) \quad \therefore \boxed{c_1 = f'(a)}$$

$$p''(x) = 2c_2 + \dots + \cancel{n(n-1)(x-a)^{n-2}}$$

$$f''(a) = p''(a) = 2c_2 \quad \therefore \quad c_2 = \frac{f''(a)}{2}$$

$$\vdots$$

$$p^{(k)}(x) = k(k-1)(k-2)\cdots 2c_k + (\quad)(x-a)$$

$$f^{(k)}(a) = p^{(k)}(a) = k(k-1)\cdots 2c_k$$

$$\therefore \boxed{c_k = \frac{f^{(k)}(a)}{k!}} \qquad k = 0, \dots, n$$

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Conclusão: Se f é derivável n vezes em $x = a$ então $p(x)$ dado acima é o único pol de grau n que satisfaz $p^{(k)}(a) = f^{(k)}(a)$, $k = 0, \dots, n$

P é chamado de Polinômio de Taylor de f no ponto a de grau n.

Notação: $p(x) = T_n(f; a)$

Exemplo: $f = \text{sen}$ $a = 0$

$$T_3(\text{sen}; 0) = \text{sen}\,0 + \text{sen}'0 \cdot (x-0) + \frac{\text{sen}''(0)}{2}(x-0)^2 + \frac{\text{sen}'''0}{3!}(x-0)^3$$

$\text{sen}\,0 = 0$

$\text{sen}'0 = 1$

$\text{sen}''0 = -\text{sen}\,0 = 0$

$\text{sen}'''0 = -\cos 0 = -1$

$$T_3(\text{sen}; 0)(x) = x - \frac{x^3}{3!}$$

$x = 0{,}1$

$$T_3(\text{sen}; 0)(0{,}1) = 0{,}1 - \frac{(0{,}1)^3}{6} \cong 0{,}0998$$

$$\text{sen}\,0{,}1 \cong 0{,}0998$$

Ex: $T_5(\exp; 0)(x) = 1 + 1 \cdot x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$

$\exp 0 = 1$

$\exp' 0 = 1$

$\exp'' 0 = 1$

$\exp''' 0 = 1$

$\exp^{(iv)} 0 = 1$

$\exp^{(n)} 0 = 1$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$

$$e = \exp 1 \simeq 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$$

$$= 2{,}7166$$

$$e = 2{,}7182818284590 45\ldots$$

Análise do Erro de Aproximação.

$$E_n(x) = f(x) - T_n(f; a)(x)$$

Teorema: Seja f $n+1$ vezes derivável em uma vizinhança do ponto a. Então

$$E_n(x) = f(x) - T_n(f; a)(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

onde c está entre a e x.

Teor. do Valor Médio de Cauchy

F e G deriváveis e (a,b) e cont. em $[a,b]$

então

$$F'(c)(G(b) - G(a)) = G'(c)(F(b) - F(a)),$$

p/ algum $c \in (a,b)$

$$F(t) = f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

$$F(x) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\cancel{(x-x)}^k = f(x)$$

$$F(a) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = T_n(f;a)(x)$$

$$\therefore \quad E_n(x) = F(x) - F(a)$$

$$F'(t) = f'(t) + \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(t)}{k!}\cdot(x-t)^k + \frac{f^{(k)}(t)}{k!}\cdot k(x-t)^{k-1}\cdot(-1)\right]$$

$$= f'(t) + \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}\right]$$

$$= \cancel{f'(t)} + \cancel{f''(t)\cdot(x-t)} - \cancel{f'(t)} + \frac{f'''(t)}{2}(x-t)^2 - \cancel{f''(t)(x-t)} + \dots$$

$$= \frac{f^{(n+1)}(t)\,(x-t)^n}{n!}$$

$$G(t) = (x-t)^{n+1}$$

$$G'(t) = (n+1)(x-t)^n\cdot(-1)$$

$$= -(n+1)(x-t)^n$$

Aplicamos Cauchy no intervalo $[a,x]$:

$$\frac{f^{(n+1)}(c)}{n!}\cdot\cancel{(x-c)^n}\cdot\left[(x-x)^{n+1}\cancel{-}(x-a)^{n+1}\right] =$$

$$= \cancel{-}(n+1)\cancel{(x-c)^n}\cdot E_n(x)$$

$$\boxed{E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\cdot(x-a)^{n+1}}$$

P/ exemplo: Calcular e com 7 casas decimais de exatidão.

$$e = \exp 1 = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!}$$

Qual deve ser o valor de n para que $E_n(1) < \frac{1}{2} 10^{-8}$.

$$E_n(x) = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!} = \frac{e^c x^{n+1}}{(n+1)!}$$

$$E_n(1) = \frac{e^c}{(n+1)!} < \frac{3}{(n+1)!} < \frac{1}{2} 10^{-8}$$

$n = 12$

e c/ 7 casas de exatidão, calculemos $1 + 1 + \frac{1}{2} + \frac{1}{3!} + \dots + \frac{1}{12!}$

$$e \simeq 2,1828182$$

Outra aplicação:

$$\arctan(1) = \frac{\pi}{4}$$

$$\therefore \pi = 4 \cdot \underbrace{\arctan(1)}_{\text{Pol Taylor}}$$

<u>Teorema:</u> $f = \exp$, sen ou cos

então $f(x) = \lim_{x \to \infty} T_n(f;0)(x)$

<u>Prova:</u> $E_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$.

<u>x fixo</u>

$$|E_n(x)| \leq \frac{|f^{(n+1)}(c)|\,|x|^{n+1}}{(n+1)!}$$

$f =$ sen, cos

$|f^{(n+1)}(c)| \leq 1$

$$|E_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

$\vdash$: $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$

existe $n_0 > |x|$

$$\frac{|x|^{n+1}}{(n+1)!} = \underbrace{\frac{|x|\cdot|x|\cdot\ldots\cdot|x|}{1\cdot 2\cdot\ldots\cdot n_0}}_{\substack{M \\ \text{constante}}} \cdot \frac{|x|\cdot|x|\cdot\ldots\cdot|x|}{(n_0+1)(n_0+2)\cdot\ldots\cdot(n_0+k)}$$

$$\underbrace{\underbrace{\frac{|x|}{n_0+1}}_{<1} \cdot \underbrace{\frac{|x|}{n_0+2}}_{<1} \cdot \ldots \cdot \frac{|x|}{n_0+k}}$$

$$< M \cdot \frac{|x|}{n_0+k} = \frac{\text{const}}{n_0+k} \xrightarrow{k\to\infty} 0$$

$f = \exp.$

$$\leq \underbrace{e^{x}}_{\text{const.}}$$

$$E_n(x) = \frac{e^{c}\cdot|x|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0$$

$\downarrow 0$

$\#$

Uma função é dita Analítica quando p/ todo x em seu domínio temos que a função coincide com o $\lim_{n\to\infty} T_n$ em uma vizinhança de x.