FRAÇÕES PARCIAIS

Funções racionais = $\left.\begin{array}{l} p(x) \\ q(x) \end{array}\right\}$ polinômios.

Exemplo: $\frac{2}{x-1} + \frac{3}{x+3} = \frac{2(x+3)+3(x-1)}{(x-1)(x+3)}$

$$= \frac{2x+6+3x-3}{x^2+2x-3} = \frac{5x+3}{x^2+2x-3}$$

$$\int \frac{5x+3}{x^2+2x-3}\,dx = \int \frac{2\,dx}{x-1} + \int \frac{3\,dx}{x+3}$$

$$= 2\ln|x-1| + 3\ln|x+3| + C$$

Exemplo: $\int \frac{2x+5}{x^2+2x-3}\,dx$

$$\frac{2x+5}{\underbrace{x^2+2x-3}_{(x-1)(x+3)}} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$\Downarrow \quad \times\,(x-1)(x+3)$$

$$2x+5 = A(x+3) + B(x-1)$$

- Coletar as pt. semelhantes de x

$$2x+5 = Ax+3A+Bx-B$$

$$= (A+B)x + 3A - B$$

$$\begin{cases} A+B=2 \\ 3A-B=5 \end{cases}$$

$$4A=7 \Rightarrow A = \frac{7}{4}$$

$$B = 2-A = \frac{8}{4} - \frac{7}{4} = \frac{1}{4}$$

- Substituir valores p/ x

$$2x+5 = A(x+3) + B(x-1)$$

$x=-3$

$$2\cdot(-3)+5 = A\cdot 0 + B(-4)$$

$$-1 = -4B \therefore B = 1/4$$

$x=1$

$$2\cdot 1+5 = A\cdot(1+3) + B\cdot 0$$

$$7 = 4A \therefore A = \frac{7}{4}$$

$$\frac{2x+5}{x^2+2x-3} = \frac{7}{4}\cdot\frac{1}{x-1} + \frac{1}{4}\cdot\frac{1}{x+3}$$

$$\therefore \int \frac{2x+5}{x^2+2x-3}\,dx = \frac{7}{4}\ln|x-1| + \frac{1}{4}\ln|x+3| + C$$

$$\boxed{\begin{array}{l} \frac{f(x)}{g(x)} = \frac{\text{linear}}{(x-k_1)(x-k_2)} = \frac{A}{x-k_1} + \frac{B}{x-k_2} \\ \int \frac{f(x)}{g(x)}\,dx = A\ln|x-k_1| + B\ln|x-k_2| + C \end{array}}$$

$\frac{f(x)}{g(x)}$ grau de $f(x)$ < grau de $g(x)$

Fç racional PRÓPRIA

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$$f(x) = g(x)\underset{\substack{\uparrow \\ \text{quociente}}}{Q(x)} + \underset{\substack{\uparrow \\ \text{resto}}}{R(x)}$$

grau $R(x)$ < grau $g(x)$

$$\Downarrow$$

$$\frac{f(x)}{g(x)} = \underset{\substack{\uparrow \\ \text{polin.}}}{Q(x)} + \frac{R(x)}{g(x)} \quad \text{PRÓPRIA}$$

$$\frac{x^3+3x}{x^2-2x-3}$$

quociente

$$\begin{array}{r|l} x^3+3x & x^2-2x-3 \\ \hline -x^3+2x^2+3x & x+2 \\ \hline 2x^2+6x & \\ -2x^2+4x+6 & \\ \hline 10x+6 & \end{array}$$

resto

$$\frac{x^3+3x}{x^2-2x-3} = x+2 + \frac{10x+6}{x^2-2x-3}$$

$$\frac{A}{(x+a)^k} \qquad \frac{Bx+C}{(x^2+bx+c)^m}$$

$A, B, C, a, b, c \in \mathbb{R}$

k, m inteiros positivos

$b^2-4c<0 \Rightarrow x^2+bx+c$ é irredutível

$$\frac{f(x)}{g(x)}$$

Como é a decomposição de $g(x)$?

CASO 1: $g(x)$ se decompõe como prod de fatores lineares distintos

$$g(x) = (x-x_1)\cdot(x-x_2)\cdot\ldots\cdot(x-x_m).$$

$x_i \neq x_j \quad i,j = 1,\ldots,n. \quad i\neq j$

$$\boxed{\frac{f(x)}{g(x)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \ldots + \frac{A_m}{x-x_m}}$$

$$\int \frac{f(x)}{g(x)}dx = \sum_{i=1}^{n} A_i \ln|x-x_i| + C$$

Exemplo. $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx$

$x^3+x^2-2x = x(x^2+x-2) = x(x-1)(x+2)$

$$\frac{2x^2+5x-1}{x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$\Downarrow$$

$$2x^2+5x-1 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$

$x=0$: $-1 = A(-1)(2)$

$-1 = -2A \quad \therefore A = \frac{1}{2}$

$x=1$: $2+5-1 = B\cdot 1\cdot 3$

$6 = 3B \quad \therefore B = 2$

$x=-2$: $2\cdot 4 - 10 - 1 = C\cdot(-2)\cdot(-3)$

$8-10-1 = 6C$

$-3 = 6C \qquad C = -\frac{1}{2}$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x} = \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

CASO 2: g se decompõe como produto de fatores lineares, alguns com repetição.

Exemplo: $\displaystyle\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx$

$$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Downarrow$$

$$x^2+2x+3 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$x=1$: $1+2+3 = A\cdot 4$

$6 = 4A \quad \therefore A = \frac{3}{2}$

$x=-1$: $1-2+3 = C\cdot(-2)$

$2 = -2C \quad \therefore C = -1$

$x=0$: $3 = \frac{3}{2} - B + 1 \quad \therefore B = \frac{3}{2} + \frac{2}{2} - \frac{6}{2} = -\frac{1}{2}$

$$x^2+2x+3 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\Downarrow'$$

$$2x+2 = 2A(x+1) + B[(x+1)+(x-1)] + C$$

$x=-1$

$\underbrace{-2+2}_{=0} = -2B + C \qquad B = \frac{C}{2} = 0 \qquad B = -\frac{1}{2}$

$$\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx = \frac{3}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + C$$

Em geral:

No denom: $(x+a)^p$ — Na decomp: $\displaystyle\sum_{i=1}^{p} \frac{A_i}{(x+a)^i}$

CASO 3: O denom. contém fatores quadráticos irredutíveis, sem repetição.

No denom: x^2+bx+c — Na decomp: $\dfrac{Ax+B}{x^2+bx+c}$

Exemplo: $\displaystyle\int \frac{3x^2+2x-2}{\underbrace{x^3-1}_{(x-1)(x^2+x+1)}}\,dx$

$$\frac{3x^2+2x-2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\Downarrow$$

$$3x^2+2x-2 = A(x^2+x+1) + (Bx+C)(x-1)$$

$A=1$; $B=2$; $C=3$.

$$\int \frac{3x^2+2x-2}{x^3-1}\,dx = \underbrace{\int \frac{dx}{x-1}}_{\ln|x-1|} + \int \frac{2x+3}{x^2+x+1}\,dx$$

$$\underbrace{\int \frac{2x+1}{x^2+x+1}\,dx}_{} + \int \frac{2\,dx}{\underbrace{x^2+x+1}_{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$\int \frac{du}{u} = \ln|u| = \ln|x^2+x+1| = \ln(x^2+x+1)$$

$$\underbrace{2\int \frac{dx}{\underbrace{\left(x+\frac{1}{2}\right)^2}_{u=x+\frac{1}{2}}+\frac{3}{4}} = 2\int \frac{du}{u^2+\frac{3}{4}}}$$

$$\int \frac{du}{u^2+\alpha^2} = \frac{1}{\alpha}\arctan\frac{u}{\alpha} \qquad 2\cdot\frac{2}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}$$

$$\frac{4}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}} = \frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$2\int \frac{dx}{x^2+x+1}$$

$$\int \frac{3x^2+2x-2}{x^3-1}\,dx =$$

$$= \ln|x-1| + \ln(x^2+x+1) + \frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$$

CASO 4: O denom. contém fatores quadráticos irredutíveis repetidos.

No denom: $(x^2+bx+c)^m$ — Na decomp: $\displaystyle\sum_{k=1}^{m} \frac{B_kx+C_k}{(x^2+bx+c)^k}$

Exemplo: $\displaystyle\int \frac{x^4-x^3+2x^2-x+2}{(x-1)(x^2+2)^2}\,dx$

$$\frac{x^4-x^3+2x^2-x+2}{(x-1)(x^2+2)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

$A=\frac{1}{3}$; $B=\frac{2}{3}$; $C=-\frac{1}{3}$; $D=-1$; $E=0$

$$\underbrace{\frac{1}{3}\int \frac{dx}{x-1}}_{\frac{1}{3}\ln|x-1|} + \underbrace{\int \frac{\frac{2x}{3}-\frac{1}{3}}{x^2+2}\,dx}_{} - \int \frac{x\,dx}{(x^2+2)^2} \;\Big\}\; -\frac{1}{2}\underbrace{\int \frac{2x\,dx}{(x^2+2)^2}}_{\frac{1}{2}\cdot\frac{1}{x^2+2}}$$

$$\underbrace{\frac{1}{3}\int \frac{2x\,dx}{x^2+2}}_{\frac{1}{3}\ln(x^2+2)} - \underbrace{\frac{1}{3}\int \frac{dx}{x^2+2}}_{-\frac{1}{3}\cdot\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}}$$

$$\therefore\; = \frac{1}{3}\ln|x-1| + \frac{1}{3}\ln(x^2+2) - \frac{1}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{x^2+2} + C$$

$$\int \frac{du}{(u^2+\alpha^2)^k} \qquad \text{subst } u = \alpha\tan t$$