TÉCNICAS ADICIONAIS DE INTEGRAÇÃO

Polinômio em 2 variáveis $p(x,y)=\sum_{i=0}^{p}\sum_{j=0}^{q} a_{i,j}\,x^i y^j$

$\int R(\operatorname{sen}x,\cos x)\,dx$, onde R é uma função racional em 2 variáveis

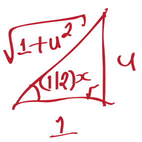
A substituição $u=\tan\frac{1}{2}x$ transforma a integral acima em uma integral da forma $\int r(u)\,du$, onde r é fç racional em 1 variável

Ex: $\int \frac{dx}{\operatorname{sen}x+\cos x} = \int \frac{2\,du}{(1+u^2)(\operatorname{sen}x+\cos x)}$

$u=\tan\frac{1}{2}x$ $\qquad \operatorname{sen}x = 2\operatorname{sen}\frac{1}{2}x\cos\frac{1}{2}x$

$x=2\arctan u$ $\qquad \cos x = \cos^2\frac{1}{2}x-\operatorname{sen}^2\frac{1}{2}x$

$dx=\frac{2}{1+u^2}\,du$



$\operatorname{sen}\frac{1}{2}x=\frac{u}{\sqrt{1+u^2}}$

$\cos\frac{1}{2}x=\frac{1}{\sqrt{1+u^2}}$

$$\operatorname{sen}x = 2\,\frac{u}{\sqrt{1+u^2}}\cdot\frac{1}{\sqrt{1+u^2}}=\frac{2u}{1+u^2}$$

$$\cos x=\frac{1}{1+u^2}-\frac{u^2}{1+u^2}=\frac{1-u^2}{1+u^2}$$

$$\operatorname{sen}x+\cos x=\frac{2u+1-u^2}{1+u^2}$$

$$\therefore \int\frac{dx}{\operatorname{sen}x+\cos x}=\int\frac{2\,du}{\cancel{1+u^2}}\;\frac{\cancel{1+u^2}}{2u+1-u^2}$$

$$=2\int\frac{du}{2u+1-u^2}=-2\int\frac{du}{u^2-2u-1}$$

$$\frac{1}{u^2-2u-1}=\frac{A}{u-a}+\frac{B}{u-b}$$

$$=\frac{A(u-b)+B(u-a)}{(u-a)(u-b)}$$

$u=a$: $\quad 1=A(a-b)$

$\therefore A=\frac{1}{a-b}$

$u=b$: $\quad 1=B(b-a)$

$\therefore B=-\frac{1}{a-b}$

$$\int\frac{du}{u^2-2u-1}=\frac{1}{a-b}\left(\int\frac{du}{u-a}-\int\frac{du}{u-b}\right)$$

$$=\frac{1}{a-b}\left(\ln|u-a|-\ln|u-b|\right)+C$$

$$=\frac{1}{a-b}\cdot\ln\left|\frac{u-a}{u-b}\right|+C$$

$$\therefore\int\frac{dx}{\operatorname{sen}x+\cos x}=-2\cdot\left(\frac{1}{a-b}\ln\left|\frac{u-a}{u-b}\right|+C\right)$$

$$=\frac{2}{a-b}\ln\left|\frac{u-b}{u-a}\right|+C$$

$$=\frac{2}{a-b}\ln\left|\frac{\tan\frac{1}{2}x-b}{\tan\frac{1}{2}x-a}\right|+C$$

a, b raízes de u^2-2u-1

$$u=\frac{+2\pm\sqrt{4-4\cdot(-1)}}{2}$$

$$=\frac{2\pm2\sqrt{2}}{2}\quad\begin{cases}1+\sqrt{2}=a\\ 1-\sqrt{2}=b\end{cases}$$

$$a-b=2\sqrt{2}$$

$$\boxed{\therefore\int\frac{dx}{\operatorname{sen}x+\cos x}=\frac{1}{\sqrt{2}}\ln\left|\frac{\tan\frac{1}{2}x-1+\sqrt{2}}{\tan\frac{1}{2}x-1-\sqrt{2}}\right|+C}$$

Já vimos: $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

$u = \arcsin x \iff x = \sin u$

$dx = \cos u \, du$

$1 - x^2 = 1 - \sin^2 u = \cos^2 u$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos u \, du}{\sqrt{\cos^2 u}} = \int \frac{\cancel{\cos u} \, du}{\cancel{\cos u}}$$

$$= \int du = u + C = \arcsin x + C.$$

Em geral:

Uma integral da forma $\int R(x, \sqrt{a^2 - x^2})\,dx$ pode ser transformada pela subst. $x = a \sin t$ numa integral da forma $\int R(a\sin t, a\cos t) \cdot a\cos t \, dt$, que pode ser resolvida pelo método acima

Ex.: $\int \frac{x\,dx}{4 - x^2 + \sqrt{4-x^2}}$

$R(x, \sqrt{4-x^2})$

$\sin t = \frac{x}{2}$

$x = 2\sin t$, $\cos t = \sqrt{1 - \frac{x^2}{4}}$

$$\int \frac{x\,dx}{4-x^2+\sqrt{4-x^2}} = \int \frac{2\sin t \cdot 2\cos t\,dt}{4\cos^2 t + 2\cos t} = \int \frac{4\sin t\,dt}{4\cos t + 2}$$

$$= \int \frac{\sin t\,dt}{\underbrace{\cos t + \frac{1}{2}}} = -\int \frac{du}{u} = -\ln|u| + C$$

$u = \cos t + \frac{1}{2}$

$du = -\sin t\,dt$

$$= -\ln\left|\cos t + \frac{1}{2}\right| + C$$

$$= -\ln\left|\sqrt{1 - \frac{x^2}{4}} + \frac{1}{2}\right| + C$$

$$= -\ln\left|\sqrt{\frac{4-x^2}{4}} + \frac{1}{2}\right| + C$$

$$= -\ln\left|\frac{1}{2}\cdot\left(\sqrt{4-x^2} + 1\right)\right| + C$$

$$= -\ln\left(1 + \sqrt{4-x^2}\right) + C_1 \quad \#$$

Outras substituições:

Integral	Substituição
$\int R(x, \sqrt{a^2 - (cx+d)^2})\,dx$	$cx+d = a\sin t$
$\int R(x, \sqrt{a^2 + (cx+d)^2})\,dx$	$cx+d = a\tan t$
$\int R(x, \sqrt{(cx+d)^2 - a^2})\,dx$	$cx+d = a\sec t$.

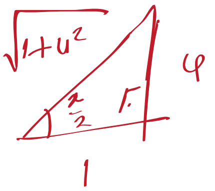
Ex : $\int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{2}{\cancel{1+u^2}} \cdot \frac{\cancel{1+u^2}}{1-u^2} du$

$u = \tan \frac{1}{2}x$ $\qquad \cos x = \cos^2 \frac{1}{2}x - \operatorname{sen}^2 \frac{1}{2}x$ $\qquad \operatorname{sen} x = \frac{2u}{1+u^2}$

$x = 2 \arctan u$

$\cos x = \frac{1}{1+u^2} - \frac{u^2}{1+u^2}$

$dx = \frac{2}{1+u^2}$



$= \frac{1-u^2}{1+u^2} =$

$\frac{\sin x}{\cos x} = \frac{2u}{1-u^2}$

$\sec x = \frac{1+u^2}{1-u^2}$

$= 2 \int \frac{du}{1-u^2} = -2 \int \frac{du}{u^2-1} =$ $\qquad \frac{1}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1}$

$A(u-1) + B(u+1)$

$\underline{u=1}$

$1 = 2B$

$B = \frac{1}{2}$

$-\frac{1}{u^2-1} = -\left(\frac{A}{u-1} + \frac{B}{u+1}\right) = -\frac{A(u+1)+B(u-1)}{(u-1)(1+u)}$

$\underline{u=-1}$

$1 = -2A$

$A = -\frac{1}{2}$

ln

$\underline{u=1}$

$1 = A \cdot 2 \therefore A = \frac{1}{2}$

$\underline{u=-1}$

$1 = B \cdot (-2) \therefore B = -\frac{1}{2}$

$\therefore 2 \int \frac{du}{1-u^2} = -\ln|u-1| + \ln|u+1| + C$

$= \ln \left| \frac{u+1}{u-1} \right| + C$

$= \ln \left| \frac{u}{u-1} + \frac{1}{u-1} \right| + C$

$\ln |$

$$-2\left(-\frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1|\right) + C$$

$$\ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{u^2-2u+1}{u^2-1}\right| + C = \ln\left|\frac{-2u}{u^2-1} + \frac{u^2+1}{u^2-1}\right| + C$$

$$\tan x \qquad \sec x$$

$$= \ln\left|\frac{u}{u+1} - \frac{1}{u+1}\right| + C$$

$$\ln\left|-(\tan x + \sec x)\right| + C$$

$$\ln\left|\tan x + \sec x\right| + C$$

$$= \ln\left|\right.$$

$$\int \sec x \, dx = \int \sec \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\underbrace{\sec x + \tan x}} \, dx = \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\,dx$